Body forces and contact forces in assemblies of magnetized pieces of matter

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When two magnets are stuck together, where do magnetic forces operate and which formulas should one apply to compute them? Such frequently asked questions do not find immediate answers in the literature on forces, mainly because the force field is obtained, by the Virtual Power Principle, as a (mathematical, vector-valued) *distribution,* not as a plain vector field, which would be more convenient for practical computation. We intend to show, in a few important cases of contact (between two linear materials with different permeabilities, between magnet and magnetizable metal, between two hard magnets, etc.), how to represent this single distribution by two vector fields, one of them borne by the bulk of the matter, the other one localized at material interfaces where discontinuities of permeability, of magnetization, etc., do occur. A general approach will then be suggested.

***Index Terms*—Magnetostatics, magnetic force, contact forces, virtual power principle.**

I. Introduction

CONSIDER a piece of matter with reluctivity ν plunged into the field of a DC coil that lies some distance away. Suppose ν insensitive to the local strain (this is to avoid for the moment the difficulties of magnetostriction), but possibly non-uniform inside the domain D occupied by the matter. One will easily find, from various sources (e.g., [1], [2], [3], etc.), that the force field inside D, or 'body force', is $^1\!/_2\, |B|^2\, \nabla\nu$. This vanishes for uniform ν. Yet the piece is attracted by the coil, so there must be a force, which cannot reside elsewhere than at the air–matter interface S, the boundary of D (Fig. 1). Indeed it can be shown (we do it in detail below) that this surface force is

$$F_S = {}^1\!/_2\,(|H_\tau|^2[\mu] - |B_n|^2[\nu])\, n, \qquad (1)$$

where n is the outward unit normal, H_τ and B_n the tangential part of H and normal part of B (both continuous across S), and $[\mu]$ and $[\nu]$ the jumps of μ and ν across the surface. (Note that $[\mu] > 0$, as a rule, and hence $[\nu] < 0$. Look at Fig. 1 for the sign conventions about the jump.)

There are several ways to prove (1). The most economical consists in taking $^1\!/_2\, |B|^2\, \nabla\nu$ 'in the sense of distributions'. I shall explain in detail what this means, but let us first see how the Virtual Power Principle (VPP) yields the magnetic force as a distribution, by its very nature.

Let v denote the velocity of a virtual motion, in which a particle sitting at point x in the reference configuration (the one for which we want to compute forces) is displaced to the point $x + tv$ at time t. We take v smooth and compactly supported (i.e., null outside some bounded region, called the support of v). Call Ψ_v, a function of virtual time only, the total magnetic energy of the system at time t, as it evolves during this virtual motion *while keeping B equal to its reference value*. Then the virtual power at time 0 is minus the derivative of $\Psi_v(t)$ at $t = 0$ (a well-known result; cf. [4] for a detailed proof), hence a linear function of v. It may happen that the virtual power has the form $\int F \cdot v$, where F is a vector field, which is then, by definition, the force field. But most often the map $v \to -\partial_{t=0}\Psi_v$ is just that: a map, linear with respect to v, with the required kind of continuity with respect to v that qualifies it as a distribution. (It's a *vector*-valued distribution, since the test functions v are themselves vector-valued.)

For instance, in the case just evoked of a piece with reluctivity ν, the linear map one finds *cannot* be written as $\int F \cdot v$, where F would be $^1\!/_2\, |B|^2 \nabla\nu$ at all points where this vector field is well defined. This would exclude S, across which both ν and $|B|^2$ are discontinuous, and thus would make us miss the surface force. The notation $^1\!/_2\, |B|^2 \nabla\nu$ will be used nonetheless for the force distribution, but it will denote a different object than F. Which object, exactly, is what we need to make clear, and the proof of (1) will come as a by-product.

This exercise will prepare us for a more difficult one, the case of hard magnets with $B = \mu_0(H + M)$ as B–H law. Several possibilities exist for how M depends on the deformation of matter. One of them was addressed in [4], where the force field 'in the sense of distributions' was found to be

$$F = -\nabla M \cdot B - {}^1\!/_2\, \mathrm{rot}(H \times B), \qquad (2)$$

where $\nabla M \cdot B$ must be understood as $\partial_i M^j B^j$ (in a system of orthonormal frames, using Einstein's convention). There is again, hidden in (2), a system of forces borne by S (the air-magnet interface, or the magnet-magnet interface where M can be discontinuous), a part of which is normal to S and the other part tangential. Formulas for these forces in the style of (1) will be derived in the full-length paper.

Finally, we shall generalize to non-linear B–H laws of the form $H = \partial_B \Psi(u, B)$, where the magnetic energy Ψ, a function of B and of the mechanical configuration u, is convex in B.

II. Distributions 101

Let us for a moment distance ourselves from electromagnetism and deal with two scalar functions f and g. (Later, they will become ν and $|B|^2$.) The test functions, smooth and compactly supported, are called φ when scalar-valued, v when vector-valued.

If f is just integrable, without more regularity, the map $\varphi \to \int f\, \varphi$ qualifies as a distribution. (All integrals of this kind, where the integration domain is left unspecified, are over all space.) But for a distribution such as the 'Dirac mass' $\varphi \to \varphi(a)$, where a is a given spatial point, there is *no* function δ_a such that $\varphi(a) = \int \delta_a\, \varphi$. Thus, distributions encompass functions and generalize them [5].

When f is not differentiable, its gradient 'in the sense of distributions' exists nonetheless. It's the distribution $v \to -\int f \operatorname{div} v$, to which one extends the notation ∇f. Then one understands $\int \nabla f \cdot v$ as $-\int f \operatorname{div} v$. A notational abuse, of course, but which makes sense: if f were differentiable all over space, one would have $-\int f \operatorname{div} v = \int \nabla f \cdot v$, indeed. Now suppose f smooth inside both D and the outer region D', but discontinuous across their common boundary S, with a jump $[f]$. Then, integrating by parts on D and D',

$$\int \nabla f \cdot v \,\widehat{=}\, -\int f \operatorname{div} v = -\int_S [f]\, n \cdot v + \int_{D \cup D'} (\nabla^s f) \cdot v, \quad (3)$$

where $\nabla^s f$ denotes the 'strong' gradient of f, well-defined in D and D', but not on S. The capped equal sign means that the integral to its left is *defined* as the one to its right.

So the vector-valued distribution denoted by ∇f in (3) can be represented by two ordinary vector fields, the almost everywhere defined $\nabla^s f$, living on 3D space, and $[f]\, n$, living on S only. We find them, as a rule, in the roles of body force and interface force, in all situations evoked here.

Now, let's try and interpret in the sense of distributions the product $g\, \nabla f$. If both f and g are smooth all over and compactly supported, one has $\int g\, \nabla f \cdot v = -\int f \operatorname{div}(gv)$. So we may take that as a definition of $g\, \nabla f$, provided $\int f \operatorname{div}(gv)$ makes sense, which requires $[g] = 0$. Then,

$$\int (g\, \nabla f) \cdot v \,\widehat{=}\, -\int_S g\, [f]\, n \cdot v + \int_{D \cup D'} g\, (\nabla^s f) \cdot v. \quad (4)$$

The constraint $[g] = 0$ is no surprise, since the product of two distributions (here g and ∇f) does not exist unconditionally. But our goal, find an interpretation of $|B|^2 \nabla \nu$ as a distribution, is thwarted, since $[|B|^2] \neq 0$ as a rule. Neither can we handle $-|H|^2 \nabla \mu$ that way, since $[|H|^2] \neq 0$ as well. Luckily, a suitable combination of these two expressions will work.

III. Proving (1)

Suppose the interface S presented as the locus of points x for which $s(x) = 0$, for some smooth real function s. (Having that *locally* is enough.) Then, the surfaces $S_\alpha = \{x : s(x) = \alpha\}$, for α in a neighborhood of zero, say $-\delta < \alpha < \delta$, make a foliation of a neighborhood of S, call it D_δ. Call d the function on D_δ defined by $d(x) = \alpha$ when x belongs to S_α. To each such point x, assign the unit vector $(\nabla d)(x)/|(\nabla d)(x)|$, hence a field n which prolongs the field of unit normals to S considered so far. Pick also two unit vectors, anchored at x, tangent to $S_{d(x)}$, mutually orthogonal, both smoothly depending on x. This way, we have a smooth system of orthonormal frames, 'adapted' to S in an obvious sense. Any smooth vector field X will have (when restricted to D_δ) an orthogonal decomposition of the form $X = X_n\, n + X_\tau$, normal part plus tangential part. When X is smooth on D and D' separately, but discontinuous across

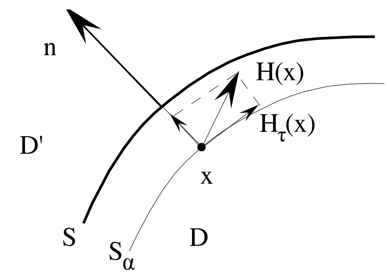

Fig. 1. The jump $[g]$ of a scalar quantity g across S is its value on the 'upstream' side of S minus its value on the 'downstream' side, as both defined by the direction of the normal field n. By convention, n goes from D to D' here. Also shown, one of the surfaces S_α of the foliation described in the text (S is S_0) and suggested, the orthogonal decomposition of the field H into normal and tangential parts.

S, one may talk of the jumps $[X_n]\, n$ and $[X_\tau]$ across S of these two parts.

Now, let's apply this to H and B, for which $[H_\tau] = 0$ and $[B_n] = 0$. Over D_δ, we have

$$\begin{aligned} |B|^2 \nabla \nu &= B_n^2 \nabla \nu + |B_\tau|^2 \nabla \nu \\ = B_n^2 \nabla \nu + |\mu H_\tau|^2 \nabla \nu &= B_n^2 \nabla \nu + |H_\tau|^2 \mu^2 \nabla \nu \\ &= B_n^2 \nabla \nu - |H_\tau|^2 \nabla \mu, \end{aligned} \quad (5)$$

since $\mu \nabla \nu = -\nu \nabla \mu$ as entailed by $\nu \mu = 1$. Thus, $|B|^2 \nabla \nu$ appears as the difference of two terms, $B_n^2 \nabla \nu$ and $|H_\tau|^2 \nabla \mu$, both of the form $g \nabla f$ with $[g] = 0$ on which we worked previously. Applying (4) to both terms yields (1).

IV. Interface terms in the case of (2)

There is room left only for a few hints about the case $B = \mu_0(H + M)$ of (2), where M is supposed to 'rotate with matter' without being affected by strain (cf. [4]). In the adapted frame system, $M = M_n n + M_\tau$, both M_n and M_τ smooth in D and D', but discontinuous, with jumps $[M_n]$ and $[M_\tau]$ across S.

The term $-{}^1\!/_2 \operatorname{rot}(H \times B)$ in (2) stands for a distribution represented by two fields. One is $-{}^1\!/_2 \operatorname{rot}^s(H \times B)$, with the strong form of the curl, the bulk force. The other one, borne by S, is expressed by one half the jump $[n \times (H \times B)]$. By the double cross product formula, this jump is $[(n \cdot B)H - (n \cdot H)B]$, which equals $n \times [n \cdot H\, n \times B]$. Substituting $\nu_0 B - M$ for H and $\mu_0(H + M)$ for B, one finds, after a short calculation,

$$[n \times (H \times B)] = \mu_0 [M_n] H_\tau - B_n [M_\tau] + \mu_0 [M_n M_\tau], \quad (6)$$

to be multiplied by $1/2$ to get the tangential part of the interface force F_S. Last, the term $-\nabla M \cdot B$ of (2) is found, by the technique of Section II, to contribute to F_S the normal field

$$\{[M_n] B_n + \mu_0 [M_\tau] \cdot H_\tau + {}^1\!/_2\, \mu_0 [|M_\tau|^2]\} n. \quad (7)$$

References

[1] G. Maugin, "The principle of virtual power in continuum mechanics: Application to coupled fields", *Acta. Mech., 35,* 1/2, pp. 1-70, 1980.
[2] F.N.H. Robinson, *Macroscopic Electromagnetism,* Oxford: Pergamon Press, 1973.
[3] G.W. Carter, "Distribution of mechanical forces in magnetised material", *Proc. IEE,* vol. 112, no. 9, pp. 1771-7, 1965.
[4] A. Bossavit, "On forces in magnetized matter", *IEEE Trans. Magn.,* vol. 50, no. 2, pp. 229-32, 2014.
[5] L. Schwartz, *Théorie des distributions,* Paris: Hermann, 1966.